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Finite two-dimensional oscillator: I. The Cartesian model

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Abstract

A finite two-dimensional oscillator is built as the direct product of two finite one-dimensional oscillators, using the dynamical Lie algebra $su(2)_x \oplus su(2)_y$. The position space in this model is a square grid of points. While the ordinary 'continuous' two-dimensional quantum oscillator has a symmetry algebra $u(2)$, the symmetry algebra of the finite model is only $u(1)_x \oplus u(1)_y$, because it lacks rotations in the position (and momentum) plane. We show how to 'import' an $SO(2)$ group of rotations from the continuum model that transforms unitarily the finite wavefunctions on the fixed square grid. We thus propose a finite analogue for fractional $U(2)$ Fourier transforms.

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1. Introduction

A two-dimensional isotropic harmonic oscillator is characterized in the familiar context of Hilbert spaces and Lie algebraic theory by the following postulates (cf [1]):

(1) There exist two (essentially self-adjoint) *position* operators, indicated by $\vec{Q} \stackrel{\text{def}}{=} (Q_x, Q_y)$, whose spectrum is the set of position values of the system.
(2) There exists a (self-adjoint and compact) *Hamiltonian* operator H, which is the generator of time evolution, satisfying the Newton–Lie or Hamilton–Lie equations

$$[H,[H,\vec{Q}]] = \vec{Q} \Longleftrightarrow \begin{cases} [H,\vec{Q}] \stackrel{\text{def}}{=} -\mathrm{i}\vec{P} \\ [H,\vec{P}] = \mathrm{i}\vec{Q}. \end{cases} \tag{1}$$

The first Hamilton equation is purely geometrical: it only defines the *momentum* operators, $\vec{P} \stackrel{\text{def}}{=} (P_x, P_y)$; the second Hamilton equation contains the oscillator dynamics. The spectrum of $\vec{P}$ is the set of momentum values of the system.

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(3) Under commutator brackets, all operators close into a Lie algebra. (In [1], q-algebras were also correspondingly postulated, and found.)

We emphasize that nothing is said in these postulates about the Lie brackets between the components of position and momentum. If one proposes $[Q_\alpha, Q_\beta] = 0 = [P_\alpha, P_\beta]$ and $[Q_\alpha, P_\beta] = \mathrm{i}\delta_{\alpha,\beta}\,\hat{1}$, then the operators will close into the two-dimensional oscillator algebra of six generators $H_6 = \mathrm{span}\{H, Q_\alpha, P_\beta, \hat{1}\}$. In [1] we showed that postulate (3) permits a small variety of Lie algebras (and q-algebras, with q in the postulate (3)), which apparently went unrecognized in oscillator physics (see, however, [2]).

The one-dimensional finite oscillator model developed in [3–6] satisfies postulate (3) with the dynamical Lie algebra $su(2)$; in this model therefore, the eigenvalues of the position, momentum and energy operators are discrete; the eigenvalue of a compact $u(1)$ central operator fixes the dimension of the $su(2)$ representation. We review the one-dimensional finite oscillator model in section 2 with $u(2)$ for dynamical algebra; its wavefunctions, involving Kravchuk polynomials, are well-known (discrete) functions of angular momentum theory. Finally, it is shown that the contraction limit of the finite oscillator, when the number and density of points grows, is the usual quantum harmonic oscillator model.

The two-dimensional finite oscillator is introduced in section 3 as a direct product of two one-dimensional finite oscillators; its position space is a square grid of points along Cartesian coordinates and its dynamical algebra is $u(1) \oplus su(2) \oplus su(2)$. Yet, as we know from the quantum oscillator, the passage from one to two dimensions entails new features, such as the enlarging of the symmetry group from $\overline{u}(1)$ to a nontrivial symmetry $\overline{u}(2)$—we use the notation with a bar to stress that it is a symmetry algebra which is physically distinct from the dynamical algebra $u(2)$ of the one-dimensional finite oscillator. In the two-dimensional finite oscillator, instead of $u(2)$, the manifest symmetry *within* the construction is only $u(1) \otimes u(1)$ (in a semidirect product with dihedral reflections D_4); this is called the *domestic* symmetry algebra [7,8]; the corresponding group consists of fractional Fourier–*Kravchuk* finite transforms along the two Cartesian axes.

A part of the symmetry transformation group of the 2D quantum oscillator (or geometric, or wave, optical waveguide [9]), $\overline{U}(2)$, can nevertheless be *imported* onto the 2D finite Cartesian oscillator system. In section 4, we adapt the concepts of domestic and imported algebras from [7, 8] to define 'angular momentum' states in the finite oscillator. Thus we can import the subgroup $\overline{SO}(2)$ of rotations by specifying the transformations undergone by those finite angular momentum states. Strictly speaking, this is not a *symmetry* of the finite Cartesian grid on a square; it is a group of unitary transformations of the wavefunctions on this grid, diagonal in mode number (energy) and which, in the limit of large grid size and density, smoothly becomes the rotation of the plane of positions (and of momenta) of the usual quantum oscillator. The fractional Fourier–Kravchuk transformations along the two grid axes, and rotations, close into a finite analogue of the 2D $U(2)$ fractional Fourier transforms [9]. Finally, we offer a discussion of the implications of this model for pixellated image analysis, amongst the conclusions in section 5.

The following paper in this series will refer to a distinct model of the finite two-dimensional oscillator, where position space consists of points on concentric, equally spaced circles. The dynamical algebra, representation and number of points will be the same, but with a specific correspondence between dynamical algebra generators and phase space observables.

2. Finite one-dimensional oscillator

In the one-dimensional finite oscillator model there is a single position operator Q, its corresponding momentum P and the Hamiltonian H. The third postulate—that these three

generators close into a Lie algebra—requires them to satisfy the Jacobi identity. Using (1), this implies that

$$[H,[Q,P]]=0 \Longleftrightarrow [Q,P]=\mathrm{i}\,G(\hat{1},H,C) \tag{2}$$

where G is a function of a unit (or generally *central*) operator $\hat{1}$, and/or the Hamiltonian H, and/or any Casimir operators C that the algebra may have. Choosing an (allowed) algebra determines a model of an harmonic oscillator. The usual 'continuous' quantum oscillator is obtained for $G=\hat{1}$ ($\hbar=1$) with the oscillator algebra $H_4=\mathrm{span}\,\{H,Q,P,\hat{1}\}$; this contains the basic Heisenberg–Weyl subalgebra $W_1=\mathrm{span}\,\{Q,P,\hat{1}\}$ of quantum mechanics.

2.1. The dynamical algebra $u(2)$

The algebra of interest to us is $u(2)=u(1)\oplus su(2)$, where the central $u(1)$ is generated by E_J, any multiple of $\hat{1}$, and where the eigenvalues of the $su(2)$ Casimir operator C are $j(j+1)$ for $j=0,\frac{1}{2},1,\ldots$. The eigenvalues serve to label and distinguish the $(2j+1)$-dimensional irreducible representation spaces of the algebra; within each of these spaces

$$-\mathrm{i}[Q,P]=G=H-E_J-\tfrac{1}{2}\hat{1}=H-(j+\tfrac{1}{2})\hat{1}\qquad E_J\stackrel{\mathrm{def}}{=}j\hat{1}. \tag{3}$$

The postulates of the finite oscillator model give a new physical interpretation to the three well-known generators of angular momentum, $\vec{J}=\{J_k\}_{k=1,2,3}$ in $su(2)=so(3)$, and of their spectra, consisting of $2j+1$ nondegenerate values:

$$Q=J_1 \longleftrightarrow \text{position } q\in\{-j,-j+1,\ldots,j\} \tag{4}$$

$$-P=J_2 \longleftrightarrow -\text{momentum } p\in\{-j,-j+1,\ldots,j\} \tag{5}$$

$$H=J_3+j+\tfrac{1}{2} \longleftrightarrow \text{Hamiltonian } h\in\{\tfrac{1}{2},\tfrac{3}{2},\ldots,2j+\tfrac{1}{2}\} \tag{6}$$

$$H-\tfrac{1}{2}=J_3+j \text{ is the mode number } n\in\{0,1,\ldots,N\}\qquad N\stackrel{\mathrm{def}}{=}2j. \tag{7}$$

The Lie brackets are the usual ones: $[J_j,J_k]=\mathrm{i}\epsilon_{jkl}J_l$ and $[J_k,E_J]=0$. Because the $u(2)$ algebra is *compact*, the corresponding oscillator model will have intrinsically *discrete and finite* position $(q|_{-N/2}^{N/2})$, momentum $(p|_{-N/2}^{N/2})$ and mode $(n|_0^N)$ values.

2.2. Position and mode eigenbases

Within the representation with $j=\frac{1}{2}N$, where the finite oscillator consists of $N+1$ equidistant points $\{-\frac{1}{2}N,-\frac{1}{2}N+1,\ldots,\frac{1}{2}N\}$, there are $N+1$ 'Kronecker' eigenstates, which we denote by

$$\begin{aligned} &Q|N,q\rangle_1=q|N,q\rangle_1,\quad q|_{-N/2}^{N/2}\\ &\vec{J}^{\,2}|N,q\rangle_1=\tfrac{1}{2}N(\tfrac{1}{2}N+1)|N,q\rangle_1. \end{aligned} \tag{8}$$

A finite oscillator of $N+1$ points also has $N+1$ energy eigenstates. To a ground state of 'zero bosons' we can add at most N '$su(2)$ bosons' to reach the highest possible state of the system. One thus introduces the second eigenbasis

$$\begin{aligned} &H|N,n\rangle_{\mathrm{H}}=(n+\tfrac{1}{2})|N,n\rangle_{\mathrm{H}},\quad n|_0^N\\ &\vec{J}^{\,2}|N,n\rangle_{\mathrm{H}}=\tfrac{1}{2}N(\tfrac{1}{2}N+1)|N,n\rangle_{\mathrm{H}}. \end{aligned} \tag{9}$$

To use freely the standard $su(2)$ results for angular momentum we should remember that the mode number $n|_0^N$ is related to the eigenvalues of J_3, $\mu|_{-j}^{j}$, through $n=j+\mu$ and

$\mu = n - \frac{1}{2}N$. To stress the abstract (model-independent) definition of these states we shall use their kets with round brackets:

$$\begin{aligned} |j,\mu)_3 &\stackrel{\text{def}}{=} |2j, j+\mu\rangle_{\text{H}} \qquad & J_3|j,\mu)_3 &= \mu|j,\mu)_3, \quad \mu|_{-j}^{j} \\ |N,n\rangle_{\text{H}} &\stackrel{\text{def}}{=} |\tfrac{1}{2}N, n-j)_3 \qquad & \vec{J}^2|j,\mu)_3 &= j(j+1)|j,\mu)_3. \end{aligned} \tag{10}$$

The relation between the 1- and 3-basis is a rotation:

$$\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi J_2} J_3 = J_1 \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi J_2} \Rightarrow |N,q\rangle_1 = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi J_2}|N, j+q\rangle_{\text{H}}. \tag{11}$$

In this connection we must also recall the abstract definition of the 'small d' Wigner functions [10]:

$$d^j_{m,m'}(\beta) \stackrel{\text{def}}{=} {}_3(j,m|\mathrm{e}^{-\mathrm{i}\beta J_2}|j,m')_3 = d^j_{m',m}(-\beta) \tag{12}$$

$$= \sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}$$
$$\times \sum_k (-1)^k \frac{(\cos\frac{1}{2}\beta)^{2j-2k+m-m'}(\sin\frac{1}{2}\beta)^{2k-m+m'}}{k!\,(j+m-k)!\,(j-m'-k)!\,(m'-m+k)!}. \tag{13}$$

The lowest mode of the oscillator is $|j,-j)_3 = |N,0\rangle_{\text{H}}$; from this, we can find the rest of the modes by means of the well-known $su(2)$ shift operators, $J_\pm \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(J_1 \pm \mathrm{i}J_2)$. Their action on the mode eigenstates is

$$J_+|N,n\rangle_{\text{H}} = \sqrt{\tfrac{1}{2}(n+1)(N-n)}|N,n+1\rangle_{\text{H}} \tag{14}$$

$$J_-|N,n\rangle_{\text{H}} = \sqrt{\tfrac{1}{2}n\,(N-n+1)}\,|N,n-1\rangle_{\text{H}} \tag{15}$$

so the ground and 'anti-ground' states obey $J_-|N,0\rangle_{\text{H}} = 0$ and $J_+|N,N\rangle_{\text{H}} = 0$. Finally, recursion of (14) displays the mode states in 'n boson' form:

$$|N,n\rangle_{\text{H}} = \left[2^n \binom{N}{n}\right]^{-1/2} J_+^n|N,0\rangle_{\text{H}}. \tag{16}$$

2.3. Contraction of the algebra $u(2) \to osc$

The limit $N \to \infty$ [11] can be seen as an algebra contraction where the shift operators (14) and (15) become the boson creation and annihilation operators of the ordinary quantum oscillator and their 'true' n boson kets:

$$A^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - \mathrm{i}\hat{P}) \qquad A^\dagger|n)_{\text{osc}} = \sqrt{n+1}|n+1)_{\text{osc}} \tag{17}$$

$$A = \frac{1}{\sqrt{2}}(\hat{Q} + \mathrm{i}\hat{P}) \qquad A\,|n)_{\text{osc}} = \sqrt{n}\;|n-1)_{\text{osc}} \tag{18}$$

$$|n)_{\text{osc}} = (n!)^{-1/2}\,(A^\dagger)^n|0)_{\text{osc}}. \tag{19}$$

The contraction holds for $n \ll N \to \infty$, when equations (14)–(16) approximate equations (17)–(19), as $\binom{n}{N} \approx 1/n!\,N^n$. This is a sequence of operators in $(N+1)$-dimensional space, whose proper convergence to an operator in the Hilbert space of square-integrable functions on the real line has been analysed by Barker [8]. When understood in this sense (and with a justifiable abuse of notation) we can write

$$\lim_{N\to\infty} (\tfrac{1}{2}N)^{-1/2} J_+ = A^\dagger \qquad \lim_{N\to\infty} (\tfrac{1}{2}N)^{-1/2} J_- = A \qquad [A, A^\dagger] = \hat{1} \tag{20}$$

while the oscillator Hamiltonian limit is obtained from $J_1^2 + J_2^2 = (N+1)H - H^2 - \frac{1}{4}$,

$$\lim_{N\to\infty} H = \lim_{N\to\infty} (J_3 + \tfrac{1}{2}(N+1)\hat{1}) = H_{\text{osc}} = A^\dagger A + \tfrac{1}{2}\hat{1} = \tfrac{1}{2}(\hat{P}^2 + \hat{Q}^2). \tag{21}$$

In this contraction, the spectra of the position and momentum operators are a growing number $N+1$ of points, placed at decreasing distance $1/\sqrt{N}$ apart, within the interval $-\sqrt{N/2}$ and $\sqrt{N/2}$ that becomes the real line; the energy spectrum of H, on the other hand, retains its ground level and unit spacing.

2.4. Kravchuk functions of the finite oscillator

The wavefunctions of the finite oscillator are the overlaps between the Q-basis and the H-basis vectors. From (10)–(12), they are given by

$$\Phi_n^{(N)}(q) \stackrel{\text{def}}{=} {}_1\langle N, q|N, n\rangle_{\text{H}} \quad (N = 2j,\ n|_0^N,\ q|_{-j}^{j}) \tag{22}$$

$$= {}_{\text{H}}\langle N,\ j+q|\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}J_2}|N, n\rangle_{\text{H}} \tag{23}$$

$$= d^j_{q,n-j}(-\tfrac{1}{2}\pi) = d^j_{n-j,q}(\tfrac{1}{2}\pi). \tag{24}$$

For the special value $\beta = \frac{1}{2}\pi$, the 'little d' Wigner functions (13) can be written as

$$d^j_{n-j,q}\left(\frac{1}{2}\pi\right) = \frac{(-1)^n}{2^j}\sqrt{\binom{2j}{n}\binom{2j}{j+q}}K_n\left(j+q;\ \frac{1}{2}, 2j\right) \tag{25}$$

with the square root of two binomial distributions and a *symmetric Kravchuk polynomial* [12] in $j+q$ [5]:

$$K_n(x;\tfrac{1}{2}, N) \stackrel{\text{def}}{=} {}_2F_1(-n, -\tfrac{1}{2}N - x; -N; 2) \ = \ K_x(n;\tfrac{1}{2}, N). \tag{26}$$

In figure 1 we show the bottom, middle and top wavefunctions of the finite oscillator.

Kravchuk polynomials are an orthogonal and complete set of functions over $N+1$ points, with summation measure given by the binomial distribution $\binom{N}{q}$. The square root of this factor is incorporated into the definition of the Kravchuk *functions* (24). When $N \to \infty$ and at positions $x = q/\sqrt{N}$, the binomial distribution becomes the well-known decreasing Gaussian integration measure for the Hermite polynomials. In this limit, as the number and density of points increase without bound, the Kravchuk functions converge to the ordinary quantum oscillator wavefunctions:

$$\lim_{N\to\infty}(\tfrac{1}{2}N)^{1/4}\Phi_n^{(N)}\left(x\sqrt{\tfrac{1}{2}N}\right) = \Phi_n^{\text{osc}}(x) \stackrel{\text{def}}{=} (\sqrt{\pi}\,2^n\,n!)^{-1/2}\mathrm{e}^{-x^2/2}H_n(x). \tag{27}$$

In [13] we have verified this limit directly, using ${}_3F_2$ hypergeometric functions.

2.5. Fractional Fourier–Kravchuk transforms

The time evolution of the finite oscillator is generated by the Hamiltonian H which rotates the Q–P plane. According to the Hamilton equations (1), this is an inner automorphism of the $su(2)$ algebra:

$$\mathrm{e}^{-\mathrm{i}\phi H}\begin{pmatrix} Q \\ P \end{pmatrix}\mathrm{e}^{\mathrm{i}\phi H} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}. \tag{28}$$

The $SO(2)$ cycle of fractional *Fourier–Kravchuk* transforms [4] of power α on $2j+1$ points is defined in terms of the oscillator evolution angle ϕ through $\alpha = 2\phi/\pi$ and

$$\mathcal{K}^\alpha \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\alpha\,(J_3+j)} = \mathrm{e}^{+\mathrm{i}\frac{1}{4}\pi\alpha}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\alpha H} \tag{29}$$

so $\mathcal{K}^{\alpha_1}\,\mathcal{K}^{\alpha_2} = \mathcal{K}^{\alpha_1+\alpha_2}$ and $\mathcal{K}^0 = \hat{1} = \mathcal{K}^4$. The 'phase correction' by $\frac{1}{4}\pi\alpha = \frac{1}{2}\phi$ implies that, both in the finite and continuous cases, the fractional Fourier(–Kravchuk) transform multiplies the nth oscillator state by the phase [14]

$$\mathcal{K}^\alpha|N, n\rangle_{\text{H}} = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi n\alpha}|N, n\rangle_{\text{H}}. \tag{30}$$

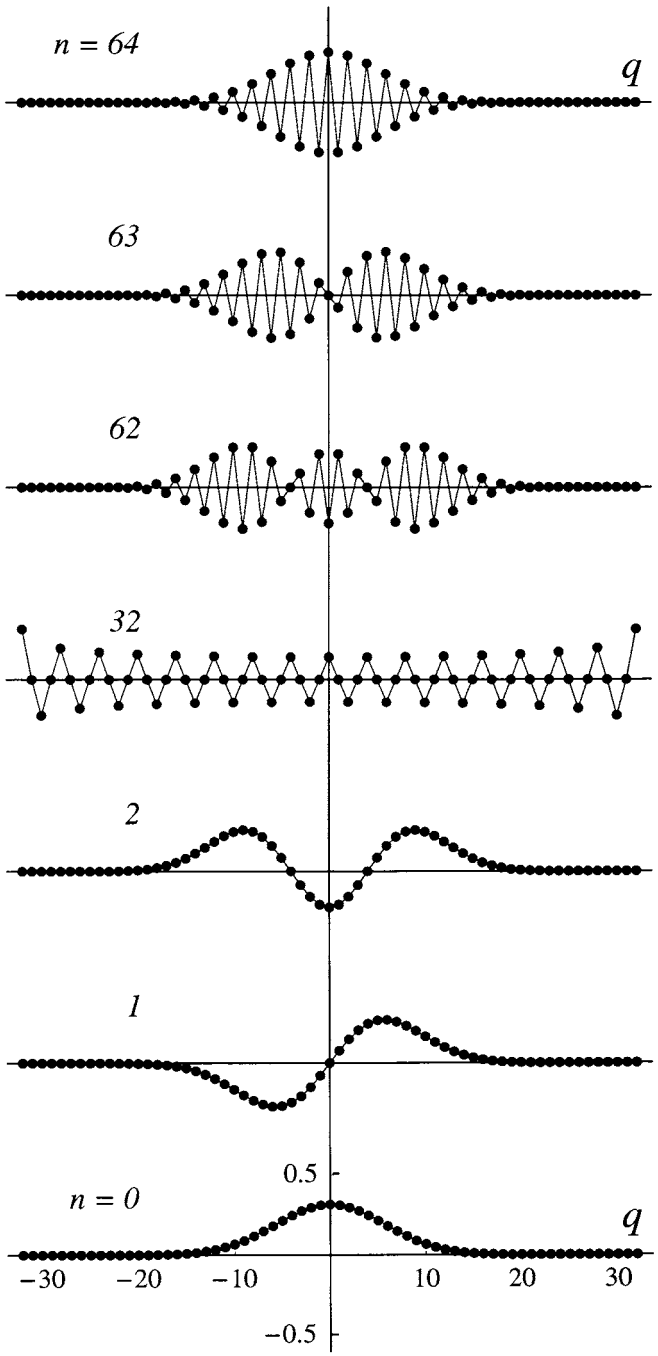

Figure 1. Kravchuk functions $\Phi_n^{(N)}(q)$ for a finite one-dimensional oscillator of $N + 1 = 65$ points, joined by straight lines. From bottom to top: the $n = 0$ ground state (root of the binomial distribution), and states of $n = 1, 2, \ldots, 32, \ldots, 62, 63, 64$ energy quanta. The last one is the highest state of the finite oscillator (the root of the binomial distribution with alternating signs).

For wavefunctions $|\psi\rangle$ on $N+1$ points of the discrete position eigenbasis, which we can expand as

$$|\psi\rangle = \sum_{q=-N/2}^{N/2} \psi^{(N)}(q)\,|N,q\rangle_1 \qquad \psi^{(N)}(q) \stackrel{\text{def}}{=} {}_1\langle N,q|\psi\rangle \tag{31}$$

the Fourier–Kravchuk transform acts on the linear combination coefficients $\psi^{(N)}(q)$ through a matrix that represents rotations around the 3-axis in the Kronecker eigenbasis of the 1-axis of positions. The kernel is given (see [10], equation 4.7(5)) through a Wigner little d function:

$$\mathcal{K}^{\alpha} : \psi^{(N)}(q) \mapsto \psi^{(N,\alpha)}(q) = \sum_{q'=-N/2}^{N/2} K^{(N,\alpha)}_{q,q'}\,\psi^{(N)}(q') \qquad \psi^{(N,0)}(q) = \psi^{(N)}(q) \tag{32}$$

$$\begin{aligned} K^{(N,\alpha)}_{q,q'} &\stackrel{\text{def}}{=} {}_1\langle N,q|\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi(J_3+j)\alpha}|N,q'\rangle_1 = \sum_{n=0}^{N} {}_1\langle N,q|N,n\rangle_{\mathrm{H}}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi n\alpha}\,{}_{\mathrm{H}}\langle N,n|N,q'\rangle_1 \\ &= \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi N\alpha}\,(-\mathrm{i})^{q-q'}\,d^{N/2}_{q,q'}(\tfrac{1}{2}\pi\alpha). \end{aligned} \tag{33}$$

The kernel is unitary $K^{(N,-\alpha)}_{q,q'} = (K^{(N,\alpha)}_{q',q})^*$, and represents the group $SO(2)$ with α modulo 4; this is evident in the before-last expression of equations (33).

The fractional Fourier–Kravchuk transforms form the $U(1)$ symmetry group of the one-dimensional oscillator (finite or standard), and the states $|N,n\rangle_{\mathrm{H}}$ have definite parity $(-1)^n$. Parity is conserved under the fractional Fourier–Kravchuk transformation, because J_3 commutes with the phase-space inversion.

Lastly, in the contraction limit $N \to \infty$, the Fourier–Kravchuk transform (with $\alpha = 1$ in (33)) becomes the ordinary integral Fourier transform. For $q = x\sqrt{\frac{1}{2}N}$, $q' = x'\sqrt{\frac{1}{2}N}$ and $j = \frac{1}{2}N$

$$\lim_{N\to\infty}\sqrt{\frac{1}{2}N}\,K^{(N,1)}_{q,q'} = \lim_{j\to\infty}\sqrt{j}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi Nj}\mathrm{i}^{q'-q}\,d^{j}_{q,q'}\left(\frac{1}{2}\pi\right) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}xx'}. \tag{34}$$

This asymptotic formula appears to be new, and again in [13] we verify this limit with the functions. The group $U(1)$ of fractional Fourier transforms is the *symmetry* group both of the finite oscillator and that of its continuous limit. Finally, *coherent states* exist in the finite model as in the standard one: they are the ground state rotated on the sphere of the $su(2)$ model [6], and in the limit they contract to the translated ground states in the phase-space plane.

3. Finite oscillator on a square grid

The simplest generalization of the one-dimensional finite oscillator of the previous section is a square grid of $(N+1)\times(N+1)$ points, with the algebra $u(1)\oplus su(2)_x\oplus su(2)_y$ of two independent and mutually commuting subalgebras (4)–(7) for the x and y directions and the same central algebra $u(1)$ of generator $E^{(x)}_J = E^{(y)}_J = \frac{1}{2}N\hat{1}$.

3.1. Position, modes and wavefunctions

From the previous section we build straightforwardly the Kronecker eigenbasis of positions (8) for a finite two-dimensional oscillator arranged in a Cartesian grid, namely

$$Q_x|N;q_x,q_y\rangle_1 = q_x|N;q_x,q_y\rangle_1, \quad q_x|^{N/2}_{-N/2} \tag{35}$$

$$Q_y|N;q_x,q_y\rangle_1 = q_y|N;q_x,q_y\rangle_1, \quad q_y|^{N/2}_{-N/2} \tag{36}$$

$$(\vec{J}^{(x)})^2|N;q_x,q_y\rangle_1 = \tfrac{1}{2}N(\tfrac{1}{2}N+1)|N;q_x,q_y\rangle_1 = (\vec{J}^{(y)})^2|N;q_x,q_y\rangle_1 \tag{37}$$

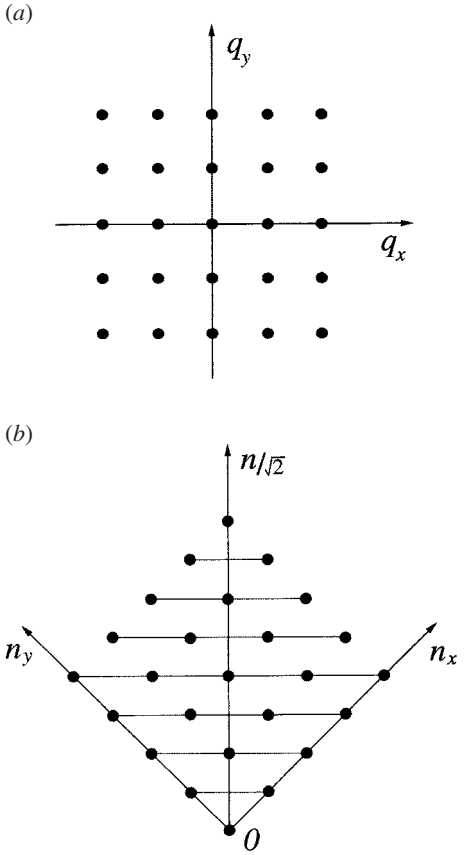

Figure 2. Two-dimensional finite oscillator (a) position space (q_x, q_y) and, (b) Cartesian modes (n_x, n_y), for 25 points ($N = 4$, $j = 2$). States with the same total number of quanta $n = n_x + n_y$ are joined with thin horizontal lines. As explained in the text, however, these states do *not* form ordinary $su(2)$ multiplets, but the $\overline{su}(2)$ group can be *imported* on their subsets.

and are shown schematically in figure 2(a).

From (16) we can build the *Cartesian modes* of the finite two-dimensional oscillator as direct products $|N; n_x, n_y\rangle_{\rm H} = |N, n_x\rangle_{\rm H_x}|N, n_y\rangle_{\rm H_y}$ obtained from the ground state $|N; 0, 0\rangle_{\rm H}$ by

$$|N; n_x, n_y\rangle_{\rm H} = \left[2^{n_x+n_y}\binom{N}{n_x}\binom{N}{n_x}\right]^{-1/2}(J_+^{(x)})^{n_x}\,(J_+^{(y)})^{n_y}|N; 0, 0\rangle_{\rm H}. \tag{38}$$

These states will satisfy (9) and (10) in the x and y components:

$$\begin{aligned} H_x|N; n_x, n_y\rangle_{\rm H} &= (n_x + \tfrac{1}{2})|N; n_x, n_y\rangle_{\rm H}, \quad n_x|_0^N \\ H_y|N; n_x, n_y\rangle_{\rm H} &= (n_y + \tfrac{1}{2})|N; n_x, n_y\rangle_{\rm H}, \quad n_y|_0^N \end{aligned} \tag{39}$$

and are shown in figure 2(b), arranged into a rhombus whose vertical axis counts the total number of quanta:

$$n \stackrel{\rm def}{=} n_x + n_y \qquad H|N; n_x, n_y\rangle_{\rm H} = (n+1)|N; n_x, n_y\rangle_{\rm H} \qquad H \stackrel{\rm def}{=} H_x + H_y. \tag{40}$$

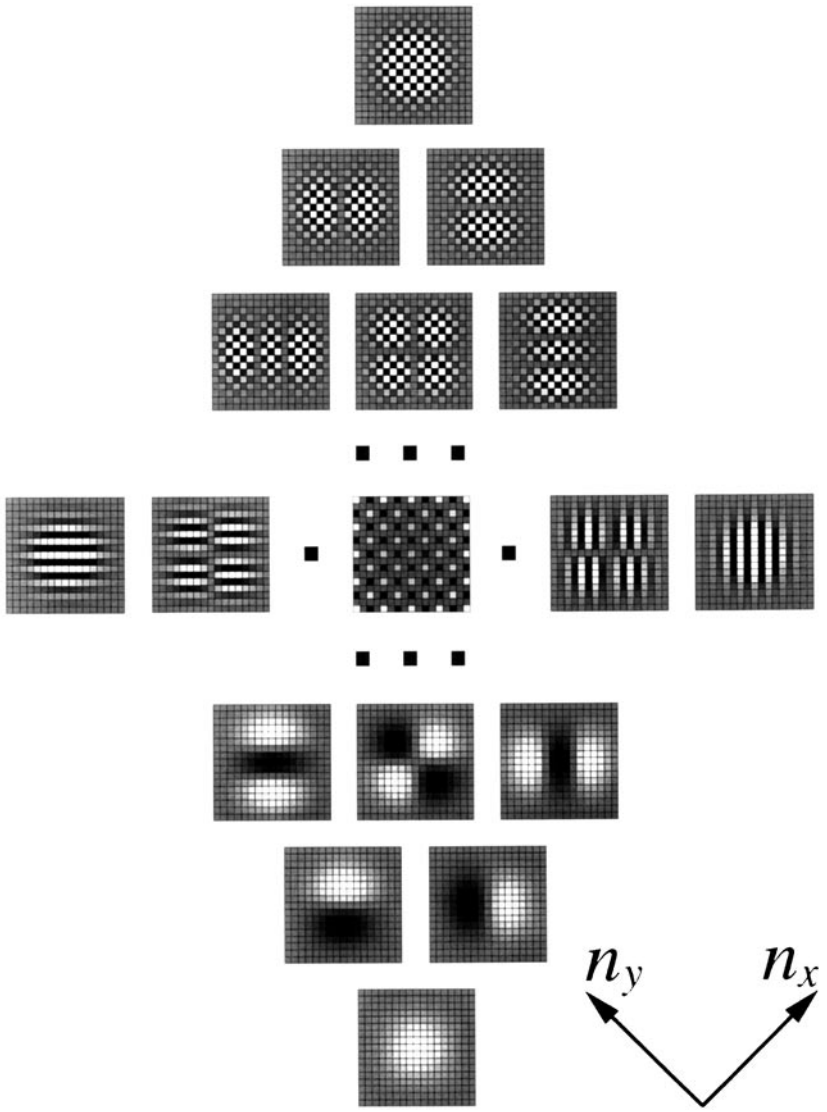

Figure 3. Cartesian eigenstates $\Phi^{(N)}_{n_x,n_y}(q_x, q_y)$ of the two-dimensional finite oscillator (17 points, $N = 16$, $j = 8$) showing density plots of selected wavefunctions. They are placed in a pattern corresponding to the rhombus of states in figure 3(*b*).

The *Cartesian mode* wavefunctions of the two-dimensional finite oscillator are simply the product of two one-dimensional ones:

$$\Phi^{(N)}_{n_x,n_y}(q_x, q_y) \stackrel{\text{def}}{=} {}_1\langle N; q_x, q_y | N; n_x, n_y\rangle_{\text{H}} = \Phi^{(N)}_{n_x}(q_x)\, \Phi^{(N)}_{n_y}(q_y). \tag{41}$$

In figure 3 we show Cartesian mode wavefunctions of a finite oscillator. The bottom states are clearly recognizable as low-energy harmonic oscillator wavefunctions, while the top states are as the former but with a checkerboard change of signs.

3.2. Domestic symmetry of the finite oscillator

Naturally, a two-dimensional finite wavefield may be subject to two independent fractional Fourier–Kravchuk transforms of powers α_x and α_y:

$$\mathcal{K}^{\alpha_x,\alpha_y} = \mathcal{K}^{\alpha_x}_{(x)}\,\mathcal{K}^{\alpha_y}_{(y)} = \mathrm{e}^{\mathrm{i}\frac{1}{4}\pi(\alpha_x+\alpha_y)}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi(\alpha_x H_x+\alpha_y H_y)} \tag{42}$$

$$\mathcal{K}^{\alpha_x,\alpha_y}\,|N;n_x,n_y\rangle_{\mathrm{H}} = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi(n_x\alpha_x+n_y\alpha_y)}\,|N;n_x,n_y\rangle_{\mathrm{H}} \tag{43}$$

$$\mathcal{K}^{\alpha_x,\alpha_y}:\psi^{(N)}(q_x,q_y)\mapsto\psi^{(N,\alpha_x,\alpha_y)}(q_x,q_y)=\sum_{q'_x,q'_y=-N/2}^{N/2}K^{(N,\alpha_x)}_{q_x,q'_x}\,K^{(N,\alpha_y)}_{q_y,q'_y}\,\psi(q'_x,q'_y). \tag{44}$$

These transforms are elements of the symmetry group $U(1)_x\otimes U(1)_y$. Following [7, 8] we call them *domestic* to the finite oscillator model. Also domestic to the finite two-dimensional Cartesian oscillator model are the $\frac{1}{2}\pi$ *rotations* of the x–y plane of position together with momentum, and inversions across the x and y axes, which form a dihedral group D_4 of outer automorphisms of the algebra $su(2)_x\oplus su(2)_y$. The parity of the state $|N;n_x,n_y\rangle_{\mathrm{H}}$ is $(-1)^{n_x+n_y}=(-1)^n$. This dihedral group, in semidirect product with the (multiply) connected group of separable x–y Fourier–Kravchuk transforms (42)–(44), constitute the domestic symmetry group of the finite two-dimensional Cartesian oscillator model.

We remark that the 'continuous' two-dimensional quantum oscillator has the larger symmetry group $U(2)$; its optical counterpart of fractional Fourier transfoms was studied in [9]. The centre $U(1)_c\subset U(2)$ is the subgroup of 'isotropic' Fourier integral transforms (cf equations (42)–(44) with $\alpha_x=\alpha_y$) and is generated by the total number operator. These and all x–y separable Fourier transforms of the continuous oscillator model are in 1:1 and limit correspondence with the Fourier–Kravchuk transforms of the finite $(N+1)^2$-point models. But the continuum model also includes continuous *rotations* of the oscillator plane by angles ϕ on the circle. With these rotations and the previous separable Fourier transforms, all $U(2)$ Fourier transformations can be generated. *Prima facie*, the finite square oscillator, whose wavefunctions can be visualized as images on a pixellated computer screen, do not seem to be amenable to continuous rotation. Yet, in the following section we shall define rotations of functions on the square.

4. Angular momentum states

In this section we shall *import* rotations from the continuum model, so that discrete functions on the pixel array can be subject to a unitary simile of rotation.

4.1. Sets of states degenerate in energy

In figure 4 we show the sets of finite oscillator states that have the same total number of energy quanta, $n\stackrel{\mathrm{def}}{=}n_x+n_y\in\{0,1,\ldots,2N\}$. They can belong to the 'lower' or 'upper' half-rhombi:

$$\big\{|N;0,n\rangle_{\mathrm{H}},\ |N;1,n-1\rangle_{\mathrm{H}},\ldots,|N;n,0\rangle_{\mathrm{H}}\big\},\quad n|_0^N \tag{45}$$

$$\big\{|N;n{-}N,N\rangle_{\mathrm{H}},\ |N;n{-}N{+}1,N{-}1\rangle_{\mathrm{H}},\ldots,|N;N,n{-}N\rangle_{\mathrm{H}}\big\},\quad n|_N^{2N}. \tag{46}$$

Only the $N+1$ states with $n=N$ quanta belong to both the lower and upper sets. Within each set, the states can be distinguished by the values (λ,κ), where

$$\lambda\stackrel{\mathrm{def}}{=}\begin{cases}\frac{1}{2}n & \text{for}\quad n|_0^N\\ N-\frac{1}{2}n & \text{for}\quad n|_N^{2N}\end{cases}\qquad\text{and}\qquad \kappa\stackrel{\mathrm{def}}{=}\tfrac{1}{2}(n_x-n_y)\in\{-\lambda,-\lambda+1,\ldots,\lambda\}. \tag{47}$$

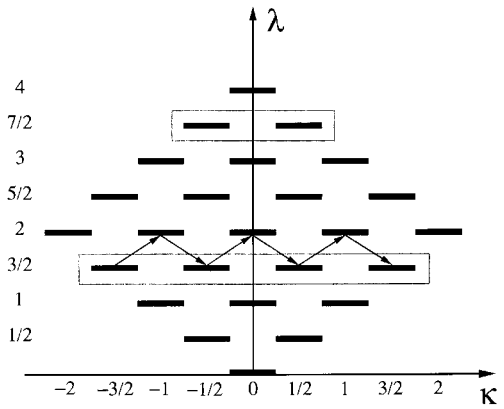

Figure 4. Eigenstates of the two-dimensional finite oscillator showing (in thin boxes) two $\overline{su}(2)$ multiplets of states degenerate in energy $n = 2\lambda = n_x + n_y$ (for $n = 3$ and 7 quanta), distinguished by $\kappa = \frac{1}{2}(n_x - n_y)$. The multiplets are not domestic because operators that can shift one state to the next, indicated by arrows (see equations (14), (15) and (48)), do not close into a Lie algebra.

It is important to realize that these sets of states, highlighted in figure 4 are *not* members of $su(2)$ multiplets. Indeed, on an $n =$ constant level, the operators that shift one state (λ, κ) to the next $(\lambda, \kappa \pm 1)$ are bilinear products of the raising and lowering operators (14) and (15), namely $J_+^{(x)} J_-^{(y)}$ and $J_-^{(x)} J_+^{(y)}$. These do raise and lower $\kappa = \frac{1}{2}(n_x - n_y)$ by 1; but because

$$[J_+^{(x)} J_-^{(y)}, J_-^{(x)} J_+^{(y)}] = \tfrac{1}{2}(J^2 + J_3^{(x)} J_3^{(y)})(J_3^{(x)} - J_3^{(y)}) \tag{48}$$

they do not close into a Lie algebra (instead, they form the algebraic basis for a *cubic* algebra [15]). Only in the limit $n_x n_y \ll N^2 \to \infty$ does the commutator (48), divided by $(\frac{1}{2}N)^2$, contract to an operator with a linear spectrum, so the set of states $n =$ constant are true $su(2)$ multiplets. It has been said that the domestic symmetry group of the finite two-dimensional oscillator enlarges from $U(1)_x \otimes U(1)_y$ to $U(2)$.

4.2. Standard $\overline{su}(2)$ multiplets of angular momentum

The symmetry group of the standard two-dimensional quantum oscillator is generated by the bilinear products of the creation and annihilation operators (17), (18), $A_x^\dagger A_x$, $A_x^\dagger A_y$, $A_y^\dagger A_x$ and $A_y^\dagger A_y$. They constitute a basis for a specific realization of the unitary algebra that we denote with bars, $\overline{u}(2)$, to emphasise that it is a physically distinct algebra from the dynamical algebra of the finite oscillator. We use the presentation of [16] to write

$$\begin{aligned} L_1 &\stackrel{\text{def}}{=} \tfrac{1}{2}(A_x^\dagger A_x - A_y^\dagger A_y) & L_2 &\stackrel{\text{def}}{=} \tfrac{1}{2}(A_y^\dagger A_x + A_x^\dagger A_y) \\ L_3 &\stackrel{\text{def}}{=} \mathrm{i}\tfrac{1}{2}(A_y^\dagger A_x - A_x^\dagger A_y) & E_L &\stackrel{\text{def}}{=} A_x^\dagger A_x + A_y^\dagger A_y. \end{aligned} \tag{49}$$

The Casimir operator turns out to be $\vec{L}^2 = \frac{1}{2}E_L(\frac{1}{2}E_L + 1)$, so $E_L = \lambda\hat{1}$ identifies the $\overline{su}(2)$ irrep $\lambda = \frac{1}{2}n$ with the set of states at the energy level $n = n_x + n_y$, which is discrete and unbounded (while in the finite model, $0 \leqslant n \leqslant 2N$).

In the Cartesian-mode basis (n_x, n_y), the generator L_1 is diagonal and has eigenvalues $\kappa = \frac{1}{2}(n_x - n_y)|_{-\lambda}^{\lambda}$, as in (47). The standard two-dimensional quantum oscillator also has the physical *orbital angular momentum* operator M, which generates rotations in the x–y plane, and which is *twice* the $\overline{su}(2)$ generator L_3:

$$M \stackrel{\text{def}}{=} Q_x P_y - Q_y P_x = 2L_3. \tag{50}$$

Hence, if $\mu|_{-\lambda}^{\lambda}$ is the spectrum of L_3 within the $\overline{su}(2)$ algebra in (49), then the spectrum of the angular momentum operator M will be the set of integer values spaced by 2:

$$m = 2\mu \qquad \mu|_{-\lambda}^{\lambda} \Rightarrow m \in \{-n, -n+2, \ldots, n\}. \tag{51}$$

We use the following two subalgebra chains [16] and their abstract kets (indicated with round brackets as before):

$$\begin{array}{ccccccccc} \overline{u}(2) & \supset & \overline{u}(1) & \oplus & \overline{u}(1) & \overline{u}(2) & \supset & \overline{u}(1)_c & \oplus & \overline{so}(2) \\ \text{Cartesian} & & \frac{1}{2}E_L + L_1 & & \frac{1}{2}E_L - L_1 & \text{polar} & & E_L & & M \\ |n_x, n_y)_1 & & \downarrow & & \downarrow & |\lambda, \mu)_3 & & \downarrow & & \downarrow \\ n_x + n_y = n & & n_x & & n_y & \lambda = \frac{1}{2}n & & n & & m = 2\mu. \end{array} \tag{52}$$

To pass from the Cartesian states to the polar states, we need their overlaps [17]. This problem is analogous to that already solved in (24)–(26); to apply those results, we replace

$$j = \tfrac{1}{2}N \to \lambda = \tfrac{1}{2}n = \tfrac{1}{2}(n_x + n_y) \tag{53}$$

$$q|_{-N/2}^{N/2} \to \kappa = \tfrac{1}{2}(n_x - n_y)|_{-n/2}^{n/2} \qquad \mu|_{-N/2}^{N/2} \to \tfrac{1}{2}m = \mu|_{-n/2}^{n/2}. \tag{54}$$

From (24) we thus find

$$|\lambda, \mu)_3 = \sum_{n_x+n_y=n=2\lambda} |n_x, n_y)_1 \, {}_1(n_x, n_y|\lambda, \mu)_3 \qquad {}_1(n_x, n_y|\lambda, \mu)_3 = d^{\lambda}_{\mu,\kappa}(\tfrac{1}{2}\pi). \tag{55}$$

Rotations of the oscillator plane by an angle θ are generated by the orbital angular momentum operator (50)

$$\mathcal{R}(\theta) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\theta M} = \mathrm{e}^{-2\mathrm{i}\theta L_3} \qquad \mathcal{R}(\theta)\,|\lambda, \mu)_3 = \mathrm{e}^{-2\mathrm{i}\mu\theta}\,|\lambda, \mu)_3. \tag{56}$$

On the L_1 eigenbasis, the action of rotations $\mathcal{R}(\theta)$ is found in exactly the same way as we found the fractional Fourier–Kravchuk transform kernel in equations(32) and (33), with n in place of N and 2θ in place of $\phi = \frac{1}{2}\pi\alpha$. However, since the bilinear generating function for the Fourier–Kravchuk transform (33) was built with phase 1 for the ground state, while for rotations the generating function has phase 1 at the mid-state in the multiplet (see (56)), the two kernels will differ by a phase in each representation:

$$\mathcal{R}(\theta)\,|n_x, n_y)_1 = \sum_{n'_x+n'_y=n} |n'_x, n'_y)_1 \, R^{(n)}_{\frac{1}{2}(n'_x-n'_y),\,\frac{1}{2}(n_x-n_y)}(\theta) \tag{57}$$

$$R^{(n)}_{\kappa',\kappa}(\theta) \stackrel{\text{def}}{=} {}_1(n'_x, n'_y|\mathrm{e}^{-2\mathrm{i}\theta L_3}|n_x, n_y)_1 \tag{58}$$

$$= \mathrm{e}^{\mathrm{i}n\theta} \, K^{(n,4\theta/\pi)}_{\kappa',\kappa} = (-\mathrm{i})^{\kappa'-\kappa} \, d^{n/2}_{\kappa',\kappa}(2\theta). \tag{59}$$

These rotation matrices are unitary and represent an $\overline{SO}(2)$ group with the parameter θ on the circle. They are diagonal for $\theta = 0, \pm\pi$ and, when $\theta = \pm\frac{1}{2}\pi$, antidiagonal with alternating signs.

4.3. Importing rotations

Now we *import* the group of rotations from the two-dimensional quantum oscillator to the two-dimensional finite model by *defining* the action of these rotations $\mathcal{R}(\theta)$ to be exactly as (57) on the finite oscillator x–y separated states $|N; n_x, n_y\rangle_{\mathrm{H}}$ in (38), as if they were the $\overline{u}(2)$ states $|n_x, n_y)_1$:

$$\mathcal{R}(\theta)\,|N; n_x, n_y\rangle_{\mathrm{H}} \stackrel{\text{def}}{=} \sum_{n'_x+n'_y=n} |N; n'_x, n'_y\rangle_{\mathrm{H}} \, R^{(n)}_{\frac{1}{2}(n'_y-n'_x),\,\frac{1}{2}(n_x-n_y)}(\theta) \tag{60}$$

$$n = \begin{cases} n_x + n_y & \text{when} \quad 0 \leqslant n_x + n_y \leqslant N \\ 2N - n_x - n_y & \text{when} \quad N \leqslant n_x + n_y \leqslant 2N. \end{cases} \tag{61}$$

Rotations acting on finite oscillator wavefunctions which are linear combinations of all $|N; n_x, n_y\rangle_{\rm H}$'s are thus block diagonal in n.

We emphasize that we have imported the transformation group $\overline{SO}(2)$, rather than any Lie algebra (whose generator, M, in (50) is not in the enveloping algebra of the finite oscillator algebra, which only contains positive integer powers of the generators). Finally we must verify that the imported $SO(2)$ rotations indeed mesh adequately with the domestic $U(1)_x \otimes U(1)_y$ fractional Fourier–Kravchuk transforms ($\mathcal{K}^{\alpha_x,\alpha_y}$ in (42) and (43)). For $\alpha_x = \frac{1}{2}\pi\phi_x$ and $\alpha_y = \frac{1}{2}\pi\phi_y$), the latter multiply the kets by the phases $\mathrm{e}^{-\mathrm{i}(n_x\phi_x+n_y\phi_y)} = \mathrm{e}^{-\mathrm{i}\frac{1}{2}n(\phi_x+\phi_y)}\mathrm{e}^{-\mathrm{i}\kappa(\phi_x-\phi_y)}$ (with $n = n_x + n_y$ and $\kappa = \frac{1}{2}(n_x - n_y)$ used as before). The first factor corresponds to the central (isotropic) Fourier transforms in $U(1)_c \subset U(2)$ which commutes with rotations. We are thus interested in the 2D Fourier–Kravchuk transforms of the special form $\mathcal{K}^{\alpha,-\alpha}$ (see (42)–(44)), whose two phases $\mathrm{e}^{\pm\frac{1}{2}\mathrm{i}n\alpha}$ will cancel. We set $\phi \stackrel{\rm def}{=} \phi_x = -\phi_y = 2\alpha/\pi$ and $\chi \stackrel{\rm def}{=} \chi_x = -\chi_y = 2\gamma/\pi$), and for each energy level $0 \leqslant n \leqslant 2N$, we find from (42)–(44) and (57)–(59) that

$$\mathcal{K}^{\alpha,-\alpha}\,\mathcal{R}(\theta)\,\mathcal{K}^{\gamma,-\gamma}|N; n_x, n_y\rangle_{\rm H} = \sum_{n'_x+n'_y=n} |N; n'_x, n'_y\rangle_{\rm H}\, D^{n/2}_{\kappa',\kappa}(2\phi, 2\theta, 2\chi) \quad (62)$$

where the Wigner 'big' D irrep functions of $SU(2)$ appear:

$$D^{n/2}_{\kappa',\kappa}(2\phi, 2\theta, 2\chi) = \mathrm{e}^{-2\mathrm{i}\kappa'\phi}\, d^{n/2}_{\kappa',\kappa}(2\theta)\, \mathrm{e}^{-2\mathrm{i}\kappa\chi}. \quad (63)$$

The D functions are well known to be the matrix elements for unitary irreducible representations of $SU(2)$.

4.4. Finite mode-angular momentum states

The consistency of the importation process allows the definition of finite oscillator states of integer *angular momentum* $m = 2\mu$ (indicated by AM). These are mathematically identical to (55), but with the physical labels of mode number n and angular momentum m. For the set of states in the lower half-rhombus ($\bigtriangledown$) in figure 2(b), we define

$$|N; n, m\rangle^{\rm AM}_{\bigtriangledown} \stackrel{\rm def}{=} \sum_{n_x+n_y=n} |N; n_x, n_y\rangle_{\rm H}\, d^{n/2}_{\frac{1}{2}(n_x-n_y),\frac{1}{2}m}(\tfrac{1}{2}\pi) \quad (64)$$

$$n \in \{0, 1, 2, \ldots, N\} \quad m \in \{\pm n, \pm(n-2), \ldots, \pm 1 \text{ or } 0\}. \quad (65)$$

Correspondingly, for the set of states in the upper half of the rhombus ($\bigtriangleup$),

$$|N; n, m\rangle^{\rm AM}_{\bigtriangleup} \stackrel{\rm def}{=} \sum_{n_x+n_y=2N-n} |N; N-n_x, N-n_y\rangle_{\rm H}\, d^{n/2}_{\frac{1}{2}(n_x-n_y),\frac{1}{2}m}(\tfrac{1}{2}\pi) \quad (66)$$

$$n \in \{N, N{+}1, \ldots, 2N\} \quad m \in \{\pm(2N{-}n), \pm(2N{-}n{-}2), \ldots, \pm 1 \text{ or } 0\}. \quad (67)$$

The largest multiplet $n = N$ belongs to both sets. The upper part contains highly oscillating wavefunctions (see figure 3) which nevertheless form multiplets of low angular momentum.

As in the two-dimensional standard quantum oscillator, energy can be divided into radial and angular contributions:

$$\begin{aligned}&\text{total quantum number } n \stackrel{\rm def}{=} 2n_r + |m| \in \{0, 1, 2, \ldots, N\}\\ &\text{angular momentum } m \in \{\pm n, \pm(n-2), \ldots, \pm 1 \text{ or } 0\}\\ &\text{'radial quantum number' } n_r = \tfrac{1}{2}(n - |m|) \in \{0, 1, 2, \ldots, \tfrac{1}{2}(n{-}1) \text{ or } \tfrac{1}{2}n\}.\end{aligned} \quad (68)$$

The radial quantum number is the degree of the Laguerre polynomial in the quantum eigenfunction. Although the finite oscillator does not decompose into 'radial and angular'

factors, we see in (25) and (26) that the degree of the Kravchuk polynomials is $j + \mu \to \frac{1}{2}(n - |m|) = n_r$. So, according to whether n is even or odd, $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ 'radial' zeros will appear on each Cartesian half-axis. The finite Cartesian harmonic oscillator angular momentum wavefunctions are obtained through the overlap of the mode-angular momentum states with the Kronecker eigenbasis of (Cartesian) position (cf equation (41)):

$$\Phi^{(N)\,\mathrm{AM}}_{n,m}(q_x, q_y) \stackrel{\mathrm{def}}{=} {}_1\langle N; q_x, q_y | N; n, m\rangle^{\mathrm{AM}} \tag{69}$$

$$= (-1)^{\frac{1}{2}(|m|-m)} \sum_{n_x+n_y=n} \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi n_y}\, d^{\lambda}_{\frac{1}{2}(n_x-n_y),\frac{1}{2}m}(\tfrac{1}{2}\pi)\, \Phi^{(N)}_{n_x}(q_x)\, \Phi^{(N)}_{n_y}(q_y) \tag{70}$$

where

$$\lambda = \begin{cases} \frac{1}{2}n & \text{when} \quad 0 \leqslant n \leqslant N \\ N - \frac{1}{2}n & \text{when} \quad N \leqslant n \leqslant 2N. \end{cases} \tag{71}$$

The total parity of $|N; n, m\rangle^{\mathrm{AM}}$ is $(-1)^n = (-1)^m$.

In figure 5 we show the angular momentum functions placed in a rhombus pattern similar to that of figure 3, but with the horizontal axis now marking angular momentum. Only the ground and top states are singlets, and at the same time Cartesian states. The functions on the boundary of the rhombus have maximal angular momentum; they exhibit $|m|$ 'diameter' zeros and suggest (in the quantum mechanical picture) that the particle circulates, staying as far from the centre as possible. The real and imaginary parts are similar, but off by a phase rotation of $\pi/2|m|$. For low n, the functions at the $m = 0$ midline ($n = 0, 2, \ldots$) show clearly the radial zeros; at $n = N$, the 'particle' seems to prefer oscillating along the diagonals. For high n, there are checkerboard sign changes, even at low m; the usual interpretation fails because the standard quantum oscillator does not have such 'anti-ground' states. These angular momentum states of the finite oscillator will transform under the imported $SO(2)$ rotations, acquiring only a phase:

$$\mathcal{R}(\theta) : \Phi^{(N)\mathrm{AM}}_{n,m}(q_x, q_y) = \mathrm{e}^{-\mathrm{i}m\theta}\, \Phi^{(N)\mathrm{AM}}_{n,m}(q_x, q_y). \tag{72}$$

In figure 6 we show the real part of one of these states as it rotates within the fixed square pixel grid.

In the contraction limit $N \to \infty$, we saw in (27) that the Kravchuk functions become Hermite functions. Now, the upper-rhombus states escape to infinity while the lower mode-angular momentum functions become the familiar Laguerre functions of the two-dimensional harmonic oscillator, because the transformation in (72) is independent of N. Hence, we obtain that

$$\lim_{N\to\infty} (\tfrac{1}{2}N)^{1/2} \Phi^{(N)\mathrm{AM}}_{n,m,\bigtriangledown}\left(\vec{q}\sqrt{\tfrac{1}{2}N}\right) = \Phi^{\mathrm{osc}}_{n,m}(\vec{q}\,)$$

$$\stackrel{\mathrm{def}}{=} \sqrt{\frac{(\frac{1}{2}(n-|m|))!}{\pi(\frac{1}{2}(n+|m|))!}}\, |\vec{q}\,|^{|m|} \left(\frac{q_x + \mathrm{i}q_y}{q_x - \mathrm{i}q_y}\right)^{\frac{1}{2}m} \mathrm{e}^{-\frac{1}{2}|\vec{q}\,|^2}\, L^{|m|}_{\frac{1}{2}(n-|m|)}(|\vec{q}\,|^2). \tag{73}$$

5. Conclusions

The two-dimensional quantum harmonic oscillator solves the Newton–Lie and Hamilton–Lie equations (1) with the Lie algebra $H_6 = \mathrm{span}\{\vec{Q}, \vec{P}\, H_x, H_y, \hat{1}\}$, while the finite oscillator satisfies the same geometric and dynamical equations with the direct sum algebra $u(1) \oplus su(2)_x \oplus su(2)_y$. But while the first has a quadratic extension that closes into the full oscillator dynamical algebra $sp(4, \mathrm{Re}\,)$, the second one does not. In particular, the symmetry

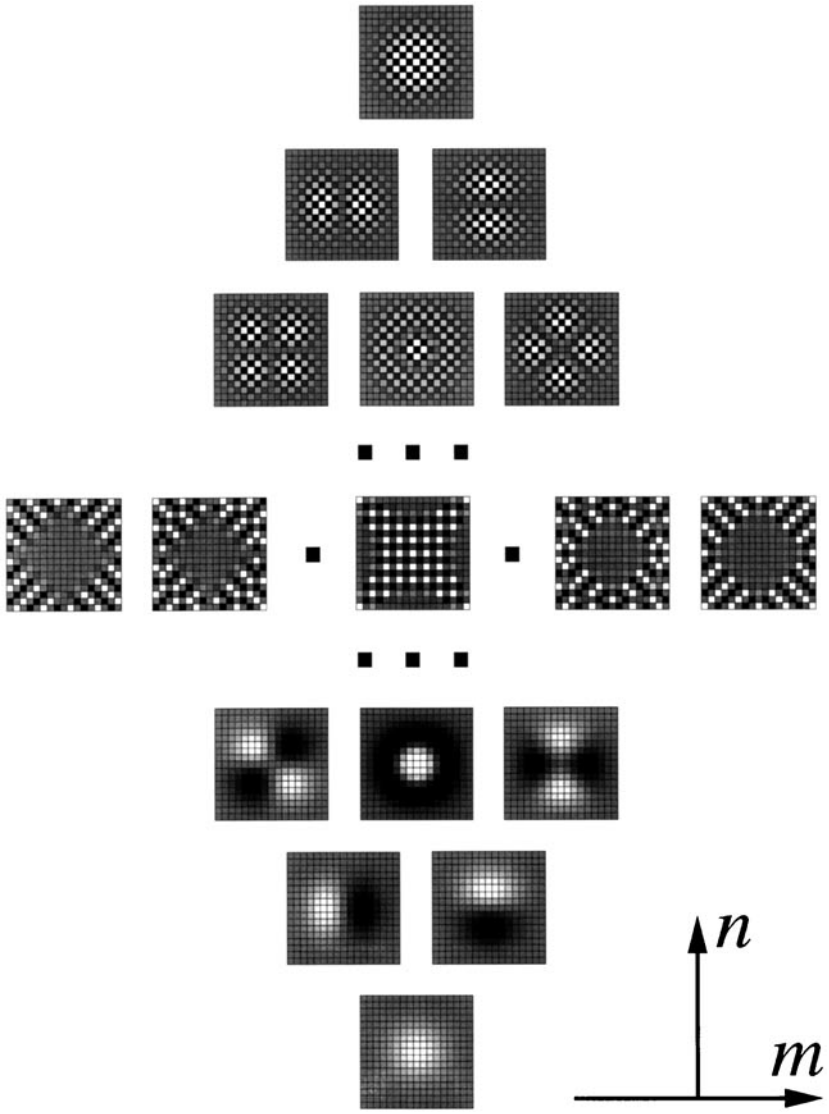

Figure 5. Angular momentum eigenstates $\Phi^{(N)\,\mathrm{AM}}_{n,m}(q_x, q_y)$ of the two-dimensional finite oscillator for $N = 17$ points. The right half of the rhombus of states ($m \geqslant 0$) shows the real part of the function (which is symmetric under $m \leftrightarrow -m$); the left half of them ($m < 0$) shows the imaginary part (which changes sign under $m \leftrightarrow -m$). In the middle ($m = 0$), the wavefunctions are real.

subalgebra $u(2) \subset sp(4, \mathrm{Re})$ of the quantum oscillator does not 'naturally' occur in the finite case.

It has not been satisfactorily explained why the symmetry groups of discrete systems can enlarge in the continuum limit. Some authors [18] have forced the states of a system into multiplets of an extraneous group by writing algebras of 'operators' that contain square roots of the Casimir operators, which do not belong to the universal enveloping algebra. Other authors working with finite models, especially those based on the very important Harper functions, whose energy spectrum is not equally spaced, import the oscillator periodic motion [7, 19].

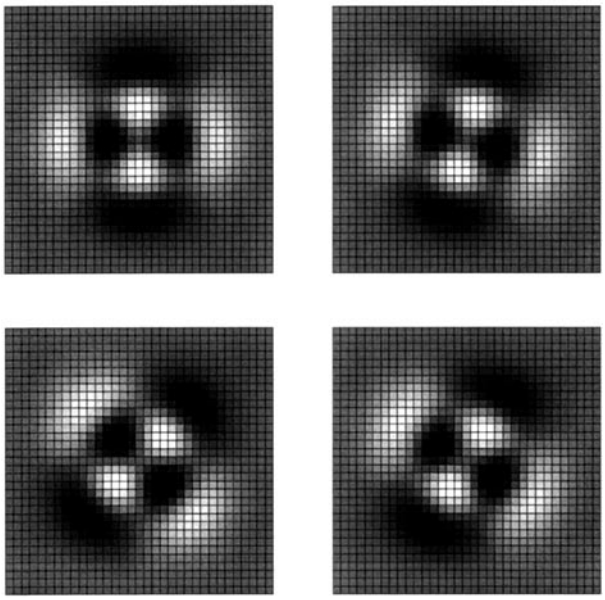

Figure 6. Clockwise from top left: rotation of the angular momentum state $\Phi_{4,2}^{(N)\mathrm{AM}}(q_x, q_y)$ in a finite two-dimensional Cartesian oscillator (33×33 pixels, $N = 32$, $j = 16$), for angles $\theta = 0^\circ, 15^\circ, 30^\circ$, and 45°.

Periodicity of motion is assured within our model of the finite oscillator because the group is compact; the Fourier–Kravchuk transform (but for a phase) is generated by the oscillator Hamiltonian. Here we have imported rotations that mesh with this domestic symmetry of a finite system. The problem of invertible rotations of pixellated images, although mentioned in the literature [20], has not been subject to a search as thorough as the closely related problem of finite fractional Fourier transformation, where one necessary factor has been to have a fast algorithm such as the FFT. We interpret the Fourier–Kravchuk transform as the proper finite analogue of the Fourier integral transform because it is a unitary representation of an $SO(2)$ group, which harbours coherent states, and which has a proper contraction limit to its continuum analogue. Similarly, $SO(2)$ rotations of a pixellated image are unitary, coherent states are constructible with well-known techniques [16], and the nature of the continuum limit is a well-posed group-theoretic problem. The group $U(2)$ of x and y oscillator evolution, and rotations, becomes a fully domestic symmetry only in the continuum limit $N \to \infty$.

It is clear that our treatment can be extended to D dimensions in Cartesian coordinates, with a domestic group $\bigoplus_{\alpha=1}^{D} U(1)_\alpha$ and an imported $SO(\mathrm{D})$ rotation group. However, we deem it more interesting to follow our investigation of the finite two-dimensional oscillator in part II of this work with the model of a finite *radial* oscillator. It could hardly have escaped the reader that the accident $su(2)_x \oplus su(2)_y = so(4)$ occurs. The dynamical algebra for the D-dimensional radial oscillator will be $so(D+2)$, and its position space pixellates the plane into a finite number of circles with radii $\rho \in \{0, 1, \ldots, N\}$, and $2\rho + 1$ 'sensor points' on each circle —with the same total of $(N+1)^2$ points. In the radial coordinate there is a finite version of the fractional Hankel integral transform (with kernel $J_m(rr')$), which are the (new) Hankel–*Hahn* transforms; these involve the discrete orthogonal dual Hahn polynomials through $so(3)$ Clebsch–Gordan coefficients.

The finite two-dimensional Cartesian and radial oscillator models correspond to two

distinct subalgebra chains of $so(4)$; in part III of this series we will explore their relation. In this way we hope to understand the separation of discrete variables as a finite counterpart of the usual separation of the oscillator in Cartesian and polar continuous coordinates [21]. With this, we can unitarily transcribe pixellated images between Cartesian and polar grids.

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